A Report On

# Study of Ocean Waves and Development of a 2-D Numerical Shallow-Water Model

By


**Dhruv Balwada** **2006A1PS448G**

At

**National Aerospace Laboratories**
**Bangalore**

A Practice School – II station of

**Birla Institute of Technology and Science, Pilani**
**Goa Campus**


**December 2009**

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**Prepared in partial fulfillment of the Practice School – II Course**

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**Goa Campus**
**December 2009**

DECLARATION

I hereby declare that the entire work embodied in the dissertation has been carried out by me and no part of it has been submitted for any degree or diploma of any institution previously.

Place: Bangalore

Date: 7/12/09

Signature of the student

**CSIR Centre for Mathematical Modelling and Computer Simulation**
(Council of Scientific & Industrial Research)
**NAL Belur Campus, Bangalore - 560 037**

C-MMACS

**Dr. P.S. Swathi**
*Scientist F*

# CERTIFICATE

This is to certify that the project entitled **'Study of Ocean Waves and Development of a 2-D Numerical Shallow-Water Model'** submitted by Mr. Dhruv Balwada is in partial fulfillment of the course BITS C412, PS II at Birla Institute of Technology & Science, Pilani. This is a bonafide record of the work done by her under my guidance and supervision at the Center for Mathematical Modeling and Computer Simulation (C-MMACS), Bangalore from 2 July 2009 to 12th December 2009.

P. S. Swm

Date: 7/12/09

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BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE

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PRACTICE SCHOOL DIVISION

Response Option Sheet

**Station:** Center for Mathematical Modeling and Computer Simulations

**Centre:** Bangalore

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**ID Number:** 2006A1PS448G

**Title of the Project:** Study of ocean waves and development of a 2-D Numerical Shallow-Water Model

| Code No. | Response Option | Response (Yes/No) |
|---|---|---|
| 1 | A new course can be designed out of this project | No |
| 2 | The project can help modification of the course content of some existing courses | No |
| 3 | The project can be hosted directly in some existing CDCs/ DCOCs/ EA courses | No |
| 4 | The project can be used in the preparatory courses like AAOC/ ES/TA and core courses | No |
| 5 | The project cannot come under any of the above mentioned options as it relates to the professional work of the host organization | Yes |

Signature of the student

Date: 7/12/09

Signature of PS faculty

Date: 7/12/09

**ACKNOWLEDGEMENT**

I would like to convey my heartfelt thanks to Dr. A R Upadhaya, Director, National Aerospace Laboratories (NAL) and Dr. Ranjan M, Head *KTMD* for giving me an opportunity to take up my project in this reputed organization.

I would also like to thank Dr. L K Maheshwari, Vice-Chancellor and Director, Birla Institute of Technology and Science, Pilani and the Practice School Division, for allowing me to pursue my PS – II at NAL.

I am grateful to our PS instructor Dr. Rakesh Mohan Jha, Scientist ‘F’ and Group Head, CEM lab, *ALD*, for his constant moral support and guidance during the project.

I am very much grateful to Dr. T. R. Ramamohan, Scientist ‘G’ and Head, CMMACS for accepting me to do work at CMMACS.

I convey my sincere gratitude to my guides Dr. K. S. Yajnik, Scientist ‘H’ and Dr. P. S. Swathi, Scientist ‘E-II’, CMMACS, for their valuable guidance, supervision and encouragement through out the project.

**BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE**

**PILANI (RAJASTHAN)**

**Practice School Division**

| | | |
|---|---|---|
| **Station** | : | National Aerospace Laboratories |
| **Centre** | : | Bangalore |
| **Duration** | : | Five and a half months |
| **Date of commencement** | : | 2nd July 2009 |
| **Date of submission** | : | 9th December 2009 |
| **Title of the Project** | : | Study of ocean waves and development of a 2-D Numerical Shallow Water model |
| **Name of the Student** | : | Dhruv Balwada |
| **ID No.** | : | 2006A1PS448G |
| **Discipline of the Student** | : | B.E. (Hons.) Chemical |
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| **Keywords** | : | numerical waves, waves, shallow water equations |
| **Project Area** | : | ocean waves |


**Abstract:**

In this report I have taken two different but important types of ocean waves; the first is the internal gravity wave and the second is the long wavelength surface wave. A comprehensive study has been conducted for the internal waves in which their reflection and refraction. The second half has been based on the shallow water equations, which describe the dynamics of long wavelength surface waves. In this part the FMS-Shallow Water Model was studied and changed to learn more about the numerics that goes into ocean modeling. The model's dynamics was studied for the cases of energy conservation schemes and potential enstrophy conservation schemes. The model's time stepping scheme was changed from leapfrog to Adams-Bashforth and a new module was added to solve for the energy-conserving scheme. This new model was tested for standard initial value test cases to check the models validity and also runs were done over idealized continental shelf topography to study the effects of wave-slope interactions. The model was finally run for the 2004 Sumatra tsunami and numerical results for the same are reported here.


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# Contents

## 0.1 Introduction

Oceans cover 71%, of the Earth's surface, 361 million sq. km. World oceans are divided into five division The Pacific Ocean, The Indian Ocean, The Atlantic Ocean, The Arctic Ocean and The Southern Ocean.

Ocean currents greatly affect the Earth's climate by transferring heat from the tropics to the polar regions, and transferring warm or cold air and precipitation to coastal regions, where winds may carry them inland. Surface heat and freshwater fluxes create global density gradients that drive the thermohaline circulation part of large-scale ocean circulation. It plays an important role in supplying heat to the polar regions, and thus in sea ice regulation. Changes in the thermohaline circulation are thought to have significant impacts on the earth's radiation budget. In so far as the thermohaline circulation governs the rate at which deep waters reach the surface, it may also significantly influence atmospheric carbon dioxide concentrations.

The ocean also has a significant effect on the biosphere. Oceanic evaporation, as a phase of the water cycle, is the source of most rainfall, and ocean temperatures determine climate and wind patterns that affect life on land. Life within the ocean evolved 3 billion years prior to life on land. Both the depth and distance from shore strongly influence the amount and kinds of plants and animals that live there

The study of the oceans and the assosciated phenomena deals with a lot of processes and their interactions. These processes can be characterized by the time and space scales over which they vary. On the rapidly varying end of the scale, there are turbulent eddies with durations of seconds and spatial scales of centimeters. At somewhat longer scales, there are propagating surface and internal gravity waves with periods of seconds to hours and wavelengths of meters to kilometers. Astronomical forces generate tides which propagate on the rotating earth as waves with periods predominantly near 12 and 24 hours and wavelengths of thousands of kilometers. At intermediate scales, there are horizontal eddies, fronts and coastal currents that vary on time scales of days to months and spatial scales of one to hundreds of kilometers. At the slowly varying end of the scale there are wind-forced and thermodynamically driven ocean currents with time scales of days to centuries and spatial scales of tens to thousands of kilometers. Transfers of momentum, heat and salt occur within the ocean and across the air-sea interface on all of these space and time scales.

### 0.1.1 Waves

Without waves, the world would be a different place. Waves cannot exist by themselves for they are caused by winds. Winds in turn are caused by differences in temperature on the planet, mainly between the hot tropics and the cold poles but also due to temperature fluctuations of continents relative to the sea. Without waves, the winds would have only a very small grip on the water and would not be

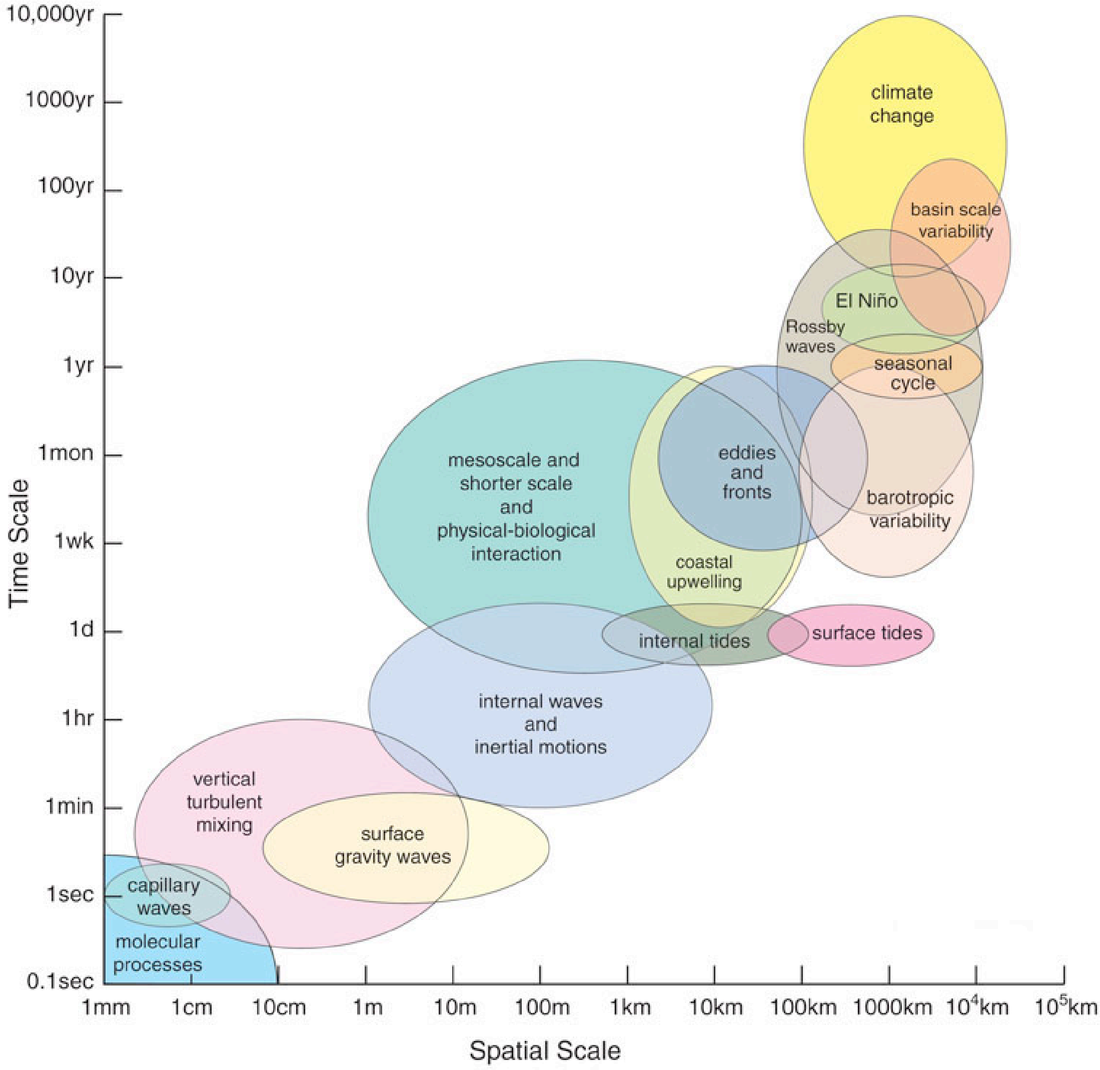


Figure 1: The processes and the scales

able to move it as much. The waves allow the wind to transfer its energy to the water's surface and to make it move. At the surface, waves promote the exchange of gases: carbon dioxide into the oceans and oxygen out. Currents and eddies mix the layers of water which would otherwise become stagnant and less conducive to life. Nutrients are thus circulated and re-used. The large ocean currents transport warm water from the tropics to the poles and cold water the other way. They help to stabilise the planet's temperature and to minimise its extremes.

## 0.2 The Basic Equations

From [2]

Velocity $\bar{u}$ in the rotating frame of reference is related to the inertial(i.e. nonrotating) velocity $\bar{u}_{inert}$ by the equation

$$\bar{u}_{inert} = \bar{u} + \bar{\Omega} \times \bar{r} \tag{1}$$

If R denotes the radius of the Earth (measured from the Earth's center to the undisturbed sea surface) and z denoted the distance measured vertically upwards from the undisturbed sea surface, then

$$r = |r| = R + z \tag{2}$$

The conservation of mass is expressed by

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \bar{u}) = 0 \tag{3}$$

or

$$\frac{D\rho}{Dt} + \rho \nabla.\bar{u} = 0 \tag{4}$$

Momentum conservation is expressed as

$$\rho \frac{D\bar{u}}{Dt} + \rho 2\bar{\Omega} \times \bar{u} = \rho[\bar{g} - \bar{\Omega} \times (\bar{\Omega} \times \bar{r})] - \nabla p + \bar{F} \tag{5}$$

where $\bar{g}$ denotes the gravitational acceleration and $\bar{F}$ represents the sum of all the other forces per unit volume acting on the fluid, including the tide producing forces as well as molecular and turbulent friction. The term $D\bar{u}/Dt$ takes the form

$$\frac{D\bar{u}}{Dt} = \frac{\partial \bar{u}}{\partial t} + \bar{u}.\nabla \bar{u} \tag{6}$$

Sea water is a complex solution of many salts. So in addition to the pressure and the temprature the salinity acts as the third state variable. The salinity is represented by S( in parts per thousand).

$$\rho = \rho(p, T, S) \tag{7}$$

Bryan and Cox suggested a form

$$\rho = A + x_1(T - B) + x_2(S - C) + x_3(T - B)^2 \tag{8}$$

$x_1$, $x_2$,$x_3$, A, B, C are determined emperically.

To close the system of equations (3), (5), (8) , for the seven unknowns p, $\rho$, $\bar{u}$ ,T, S(where $\bar{u}$ has three components) we require two more conservation equations for T and S.

Conservation of internal energy is given by

$$\frac{D(\rho c_v T)}{Dt} = \nabla.(k_T \nabla T) + Q_T \tag{9}$$

where $c_v$is the specific heat at constant volume, $k_T$ the thermal conductivity and $Q_T$ represents all sources and sinks of heat due to shearing motions and all heat transfer terms through the sea surface.

Conservation of salt is represented by

$$\frac{DS}{Dt} = \nabla.(K_S \nabla S) + Q_S \tag{10}$$

where $K_S$ denotes the coefficient of molecular diffusion of salt and $Q_S$ includes all sources and sinks due to phenomena such as ice melting and formation, precipitaion and evaporation.

In the ocean high frequency acoustic waves arise due to compressibility of sea water, and much lower frequency internal, inertial and planetary waves which exist because of gravitational and rotational forces. We are not interested in the acoustic waves for the purpose at hand so we neglect the high frequency waves. This is achieved by assuming the ocean to be incompressible, which can be represented by

$$(\partial \rho / \partial p)_\eta = 0 \tag{11}$$

where $\eta$ represents the entropy. Since in reality $(\partial \rho / \partial p)_\eta = 1/c_s^2$, the assumption means that we are assuming the speed of sound propagation to be infinite. Also as most of the processes are faster than diffusion we will assume that the ocean is non diffusive. This implies that density is constant along a particle path : $D\rho/Dt = 0$.

In view of this, (4) reduces to the continuity equation

$$\nabla.\bar{u} = 0 \tag{12}$$

Thus for an incompressible, non diffusive fluid the momentum equation along with (12) form a closed set for $\bar{u}$, p , $\rho$.

We will work with the unforced and non dissipative momentum equation, which is known as the Euler equation for a rotating fluid

$$\rho \frac{D\bar{u}}{Dt} + \rho 2\bar{\Omega} \times \bar{u} + \nabla p - \rho \bar{g} = 0 \tag{13}$$

This equation along with (12) will be referred to as the adiabatic equations, since they include neither friction nor heat or salt diffusion or sources.

At the rigid boundaries the normal component of velocity must vanish

$$\bar{u}.\bar{n} = 0 \tag{14}$$

At the disturbed sea surface $z = \eta(x, y, t)$,we require continuity of stress and displacement. These conditions can be represented as

$p_{ocean} = p_{atmosphere}$ at $z = \eta$

and $D(z-\eta)/Dt = 0$ at $z = \eta$; this condition can also be represented as $w = D\eta/Dt$.

## 0.3 Scales

Usually there are many different scales, either external ones, imposed by bottom topography or variation of wind stress, or internal ones typical of wave lengths generated by the system itself.

Vertical length scales(H) include:

1. the water depth h (water surface level is $h + \eta$)
2. the thickness $\delta$ of boundary layers in the bottom or at the free surface. They are generated by viscosity and possibly influenced by the earth's rotation. As they are determined by the flow, the boundary layer thickness is an internal scale. In rivers, lakes and coastal seas the, the boundary layers are usually thicker than the water depth, so actually the whole layer of water is within the boundary layer. In that case, the depth h is the relevant vertical scale. On the other hand, in very deep water(eg. in the ocean), the boundary layers may be very thin in comparison with the water depth; then the water depth is still the relevant vertical scale for overall motion.
3. the variation of bottom level : depth of channels, height of sand banks
4. the variation of water level (i.e. wave amplitude); this is again an internal scale, if it is not directly imposed from the outside.

Horizontal length scales(L) may also come in various kinds:

1. physical dimensions of the basin: width of a river on estuary, length of a hrbour basin
2. horizontal scales of bottom topography : width of channels, dimensions of sand banks
3. the distance over which the external forces vary significantly : variation of wind stress and atmospheric pressure in a storm
4. the wavelength, which is an internal scale determined by other factors such as the frequency of tidal forces and the dimensions of the basin.

For some purposes it is useful to introduce scales for other variables as well. For velocity this usually an internal scale U. This will generally be the typical horizontal flow velocity. For time there may be an external scale, such as the tidal period, or an internal one, such as the the wave period of a long shallow-water wave.

With these scales, three important dimensionless numbers may be formed, which determine the the importance of several terms in the equations and, therefore, the type of flow:

1. Reynolds number : $Re = UH/\nu$ : ratio of inertia to viscous terms
2. Rossby number : $Ro = U/fL$ : ratio of inertia to coriolis terms
3. Froude number : $F = U/(gh)^{1/2}$ : ratio of inertia to pressure-gradient terms

In the case of turbulence the Reynolds number is often in the range of 100 to 1000. Depending on the length scale, the Rossby number is of the order of 1 or above ( rivers and coastal regions) to $10^{-3}$ (large-scale ocean currents). This means that the Coriolis acceleration is of minor importance in small-scale flows but dominated in large scale flows. The Froude number is usually small: it rarely exceeds 0.1 to 0.2. However this is not small enough to neglect the inertia terms.

## 0.4 Boussinesq and linear approximations

If x,y have typical scales L, and the horizontal velocity component u, v are of the order U, each of the first two terms in (12) is of the order of U/L. However they will not usually cancel one another, which means that the third term in w als has to be of the same order. Assume the vertical length scale to be h, then the consequence is that the vertical velocity component is of the order $w = Uh/L$, i.e. it is smaller than the horizontal components by the same factors as the length scales.

The second step is made by considering the vertical momentum equation of (13). Using the length and velocity scales, we can roughly estimate all terms except the pressure gradient. Due to scaling of w, all advective terms turn out to be of comparable magnitude. We cannot neglect the vertical advection terms. Comparing everything with the gravitational term $\rho g$:

| local acceleration | Advective terms |
|---|---|
| $Uh/gTL$ | $U^2h/gL^2$ |

where it has been assumes that $T = L/(gh)^{1/2}$. With the values of the dimensionless numbers from the previous section we can say that all terms are small relative to the gravitational acceleration. Only the pressure gradient remains to balance it, and simplifies the z-coordinate equation of (13) to the hydrostatic pressure distirbution

$$\frac{\partial p}{\partial z} = -\rho g \tag{15}$$

Applying the boundary condition at the top and botto,, we get

$$p = g \int_{z}^{h} \rho dz + p_a \tag{16}$$

The density is taken to be constant over depth as we will be using depth averaged equations here. So we get

$$p = \rho g(h - z) + p_a \tag{17}$$

This can be use to determine the pressure gradients.

An important quantity is the Brunt-Vaisala Frequency N(z), defined as

$$N^2 = -\frac{g}{\rho_o}\frac{\partial \rho_o}{\partial z} \tag{18}$$

For stable static equilibrium in an incompresible fluid, $N^2 > 0$ for all z, or equivalently $\partial\rho_o/\partial z < 0$ for all z. The quantity N, which is also known as the stability or buoyancy frequency, is the natural(angular) buoyant frequency of of oscillation assosciated with small-amplitude, simple-harmonic motion of a neutrally buoyant element of fluid moving up and down along a line parallel to $\bar{g}$.

To describe the motion which represent departures from the static state, we introduce the perturbation pressure and density defined by the equations

$p = p_o + p'$, $\rho = \rho_o + \rho'$

Putting them in equation (12), (13) and ignoring higher order terms leads to

$$\frac{D\rho'}{Dt} + w\frac{\partial \rho_o}{\partial z} = 0 \tag{19}$$

$$\rho_o \frac{D\bar{u}}{Dt} + \rho_o 2\bar{\Omega} \times \bar{u} + \nabla p' - \rho'\bar{g} = 0 \tag{20}$$

Thus for motion in a stratified, incompressible fluid, it is only the density perturbation field that is important in the gravity term. The fluid is now acted upon by a reduced gravitational (or buoyancy) acceleration $g' = \rho' g/\rho$ and a modified pressure $p' = p - p_o$. The buouyancy force $-\rho' g$ pulls back any particle displaced from its original equilibrium level. This internal restoring force gives rise to internal wave oscillations in a stably stratified fluid.

We can write $\rho = \rho_o(1 - \rho'/\rho_o)$

Since $\rho'/\rho_o \ll 1$ uniformly throughout the ocean [in fact $\rho'/\rho_o = O(10^{-3})$], the denisty perturbation produces a small correlation to the inertia and Coriolis accelerations in Euler's equations. However as mentioned above, the denisty perturbation is important in the buoyancy term. We will define a new term $\rho_*$, which is the depth averaged value of $\rho_o(z)$.

The linearisation of the equations (19, (20) about the static state are obtained by neglecting the products of all perturbation quantities $\bar{u}$, $p'$, $\rho'$. Linearistaion is valid for wave motions of infinitesimal amplitude, which implies that particle speed is much smaller than phase speed of the wave.

The above two approximations can be combined to give the linearised Boussinesq equations, which have equation

$$\frac{\partial \bar{u}}{\partial t} + 2\bar{\Omega} \times \bar{u} + \frac{1}{\rho_*}\nabla p' - \bar{g'}_* = 0 \tag{21}$$

where $\bar{g'}_* = (\rho'/\rho_*)\bar{g}$.

# 1 Internal Waves

The waves most commonly seen in the ocean are surface waves. The waves generally occur at the interface of two mediums, the surface wave is the transfer of energy at the interface of atmosphere and the ocean. Similarily another class of waves are the internal waves. They occur generally at the interface of layers of the ocean. Layers across which temprature or salinity changes are so rapid that they cause sharp changes in density. Some places where these layers are found are:

1. Where the river(fresh water) meets the ocean. A layer of fresh water is formed over the denser saline sea water. This causes to form a two layer system allowing for internal waves to propagate between them. Such conditions also occur in fjords where it rains a lot and also the ice from the montains melts and provides fresh water.
2. The interface of pynocline and the deep waters. The ocean can generally be divided into 3 layers. The first being the surface mixed layer where the thermodynamics quantities are almost contstant throughout due to the wind mixing on the top. The next is the pynocline which is a strong vertical density gradient. This maybe caused by temprature gradient(thermocline) or due to salinity(halocline). The next layer is the deep ocean where there is no strong vertical gradient.

These are very important in the ocean. They are intimately related with the vertical fluxes of heat, salt, and momentum. The mixing produced by them affects the chemical and ecological ocean systems. The most important regions in which the internal waves act are near the slopes and regions where they break and cause mixing. Thus the reflection and breaking of these waves are important processes to be understood.

## 1.1 Equations

In a stratified but non-rotating fluid the only restoring force is due to buoyancy. Oscillations about the equilibrium state are called internal gravity waves, indicating both their dynamic origin and the fact that occur within the interior of the fluid. Refrence [2]

Starting with the linearised equations

$$\rho_o(\partial_t \bar{u}) = -\nabla p + \rho \bar{g} \tag{22}$$

$$\rho_t + w\rho_{oz} = 0 \tag{23}$$

$$\nabla.\bar{u} = 0 \tag{24}$$

It will be convenient to split the velocity vector into two parts

$\bar{u} = \bar{u}_h + w\hat{z}$

where $\bar{u}_h$ is a two-dimensional vector in a plane normal to the direction of graviy vector and w is the velocity component anti parallel to the gravity vector.

The linearised equation then become

$$\rho_o \partial_t \bar{u}_h = -\nabla_h p \tag{25}$$

$$\rho_o w_t = -p_z - \rho g \tag{26}$$

$$\rho_t + w\rho_{oz} = 0 \tag{27}$$

$$\nabla_h.\bar{u}_h = -w_z \tag{28}$$

where $\nabla_h$is the gradient operator normal to $\bar{g}$.

Differentiating (26) with respect to time and substituting $\rho_t$ gives

$$\rho_o(w_{tt} + N^2 w) = -p_{zt} \tag{29}$$

where $N^2 = -g\rho_{oz}/\rho_o$is the stability frequency. eliminating $\bar{u}_h$ between (25) and (28) yields another equation for w and p

$$\rho_o w_{zt} = \nabla_h^2 p \tag{30}$$

Finally elimination of p from (29) and (30) yeilds an equation for w alone

$$\nabla^2 w_{tt} + N^2 \nabla_h^2 w - \frac{N^2}{g} w_{ztt} = 0 \tag{31}$$

For simplicity we assume that the buoyancy frequency is independent of the depth i.e. $dN^2/dz = 0$. Then we assume that there exists plane wave solutions of the form

$$\begin{bmatrix} \bar{u}_h \\ w \\ p \end{bmatrix} = \begin{bmatrix} U_h \\ W \\ \Pi \end{bmatrix} exp[i(\bar{k}.\bar{x} - \omega t)]$$

Substitution in (31) yields the dispersion relation

$$k^2 - \frac{N^2}{\omega^2}(k_1^2 + k_2^2) + ik_3\frac{N^2}{g} = 0 \tag{32}$$

the prescence of an imaginary coefficient in the dispersion relation indicates that atleast one of the components of $\bar{k}$ must be complex.

The condition; $N^2$ is constant means that

$$\rho_o(z) = \rho_o(0)exp(-N^2 z/g) \tag{33}$$

On substituting $k_3 = Re(k_3) + iIm(k_3)$ into the dispersion relation and seprating into real and imaginary parts, we find that

$$Im(k_3) = -N^2/2g \tag{34}$$

as well as the following real dipsersion relation

$$(k_1^2 + k_2^2)(\frac{N^2}{\omega^2} - 1) = [Re(k_3)]^2 + (\frac{N^2}{2g})^2 \tag{35}$$

Writin $k_1^2 + k_2^2 = k_h^2$ and $k_h^2 + [Re(k_3)]^2 = k^2$, the dispersion relation may be written explicitly for $\omega^2$

$$\omega^2 = \frac{N^2 k_h^2}{[k^2 + (N^2/2g)^2]} \tag{36}$$

The frequency ranges from $\omega = 0$ for upward phase propagation ($k_h = 0$) to $\omega = N$ for horizontal phase propagation ($k_3 = 0$) at high wavenumbers ($k_h \gg N^2/2g$).

For short waves, such the $k \gg N^2/2g$, eqn (36) may be simplified to

$$\omega = Ncos\theta \tag{37}$$

where $|\theta| \leq \pi/2$ is the angle between the horizontal and the wavenumber vector. The short wave approximation is the same as the Boussinesq approximation, which states that variations in density have a negligible influence in the inertia terms. The phase velocity under this approximation is

$$\bar{c} = \frac{Ncos\theta}{k^2}\bar{k} \tag{38}$$

and the group velocity is

$$\bar{c}_g = \frac{Nk_3^2}{k_3 k_h}[k_1, k_2, (\frac{-k_h^2}{k_3^2})k_3] \tag{39}$$

We see that $\bar{c}_g.\bar{c} = 0$ and the energy propagates normally to the phase. This situation implies that phase and energy propagation are always in the opposite vertical directions.

## 1.2 Reflection of Internal Gravity Waves

The reflection is a very important phenomena as depending on whether the waves are forward or backward propagating after reflection determines the mixing that is produced by them.

To start the analysis we will first derive equations for individual velocity components from equation (22), (23)

$$\rho_o U = k_1 \Pi/\omega \tag{40}$$

$$\rho_o V = k_2 \Pi/\omega \tag{41}$$

$$\rho_o W = -\omega k_3 \Pi/(N^2 - \omega^2) \tag{42}$$

### 1.2.1 Reflection by a solid wall

We will restrict the analysis to constant N, and only reflection from plane surfaces will be considered.

Let the reflecting surface be

$$z - \alpha x = 0 \tag{43}$$

The boundary contains the vectors $s = (1, 0, \alpha)$ and has its outward normal along $n = (\alpha, 0, 1)$. The wavenumber of the incident wave is restricted to stay in the (x,z) plane. From the dispersion relation we get

$$\frac{k_3^2}{k_1^2} = (N^2 - \omega^2)/\omega^2 = R^2 \tag{44}$$

we shall take R>0. Since c and $c_g$ are orthogonal, it follows that the slope of the rays (in the direction of $\bar{c}_g$) has magnitude 1/R .

The incident wave is a plane wave for which the $\bar{c}_g$ points towards the reflecting boundary. Using the subscript i for incident we have

$W_i = W_o exp[i(k_1 x + k_3 z - \omega t)]$

For the reflected wave $\bar{c}_g$points away from the reflecting boundary. We write

$W_r = AW_o exp[i(l_1 x + l_3 z - \sigma t)]$

where A is the amplitude reflection coefficient. On the boundary the normal vector must vanish $\bar{U}.\bar{n} = 0$,which takes the form

$W_i + W_r - \alpha(U_i + U_r) = 0$

Using eqns together with the assumed forms for $W_i$and $W_r$, this condition becomes

$(1 + \alpha k_3/k_1)exp\{i[(k_1 + \alpha k_3)x - \omega t]\}$

$+A(1 + \alpha l_3/l_1)exp\{i[(l_1 + \alpha l_3)x - \sigma t]\} = 0$

which must hold for all values of x and t. This is possible only is

$$\sigma = \omega$$

and $l_1 + \alpha l_3 = k_1 + \alpha k_3$, i.e. $\bar{l}.\bar{s} = \bar{k}.\bar{s}$

These conditions impose that the frequency of incident and reflected wave must be the same. Also the second relation, imposes a conservation of crests along the boundary by equating the incident and reflected wavenumber components parallel to the wall. Since $\omega = \sigma$ both the waves follow the same dispersion relation.

$k_3 = \pm Rk_1$; $l_3 = \pm Rl_1$

Opposite signs must be chosen, because if we look at the case of choosing same signs, we see that both the incident and reflected waves travel in the same directions. This cannot be the case.

So,

$k_3 = Rk_1$; $l_3 = -Rl_1$

$k_3 = -Rk_1$; $l_3 = Rl_1$

We get relations between the reflected and incident wave characterstics as:

for horizontal wavenumber $l_1 = \frac{(1 \pm \alpha R)}{(1 \mp \alpha R)} k_1$

for vertical wavenumber $l_3 = -\frac{(1 \pm \alpha R)}{(1 \mp \alpha R)} k_3$

for amplitude relation $A = -\frac{(1 \pm \alpha R)}{(1 \mp \alpha R)}$

When $\alpha R < 1$, i.e., when the magnitude of the bottom slope is less than that of the rays, $l_1$and $k_1$ are of the same sign and $l_3$ and $k_3$ of opposite signs. In that case the group velocity reflects about the vertical axis, the reflected and incident rays make equal angles about the vertical. The bottom is said to be horizontally transmissive for this case for the upper frequency range for which

$0 \leq \alpha < 1$,i.e. $(\alpha^2 N^2)/(1 + \alpha^2) < \omega^2 \leq N^2$

For $\alpha R > 1$ on the other hand, reflection takes place along the horizontal axis. For such slopes the rays are said to be horizontally reflective. This coves the lower frequency range

$0 \leq \omega^2 < (\alpha^2 N^2)/(1 + \alpha^2)$

### 1.2.2 Reflection by the sea surface

At the sea surface $z = \eta(x, y, t)$

From the boundary conditions we get

$p_o(0) + \eta p_{oz}(0) + ... + p(x, y, 0, t) + \eta p_z(x, y, 0, t) + ... = p_a$

$w(x,y,0,t) + \eta w_z(x,y,0,t) + ... = \eta_t + ...$

The nonlinear terms in $D\eta/Dt$ are dropped on the basis of the linearisation assumption, that the phase speed is much greater than the particle speed.

Using the hydrostatic pressure gradient $p_{oz}$

$p_o(0) + p(x,y,0,t) - \rho_0 g\eta = p_a$

$w(x,y,0,t) = \eta_t$

Eliminating $\eta$ from these two equations

$p_t - \rho_o g w = 0$ at z=0

From this $\rho_o$ may be eliminated to get

$$(\partial_{tt} + N^2)p + gp_z = 0 \tag{45}$$

at z=0.

In a strongly stratified fluid $\omega^2 < N^2$ and under the Boussinesq approximation $N^2 p \ll gp_z$.Hence the equation reduces to

$$p_z = 0$$

Which is equivalent to w=0. This implies, under these conditions the sea surface acts as a rigid wall with $\alpha = 0$.

So the wave characterstics reduce to

$l_1 = k_1$; $l_3 = -k_3$; $A = -1$

We see that the phase changes by $180^o$ upon reflection.

The reflection rules apply to cases where N varies with z only provided the scale depth of the variation of N is well in excess of the vertical wavelength; $N_z \ll Nk_3$.

## 1.3 Refraction of Internal Gravity Waves

We will do the analysis of refraction based on ray theory, which has been explained in the appendix.

Consider waves emanating from a point z=-d in a strongly stratified fluid in which N(z) is a monotonic increasing function of z. As the medium varies only in z direction then according to ray theory $\omega$, $k_1$ and $k_2$are invariant along a ray. Let us choose $k_2$ to be zero for simplicity. The dispersion relation then reduces to

$$\frac{k_1^2}{k_3^2} = \frac{\omega^2}{(N^2 - \omega^2)} \tag{46}$$

and the components of group velocity are given by

$$\bar{c}_g = [k_1, 0, -k_3(k_1^2/k_3^2)](N^2 - \omega^2)/\omega k^2 \tag{47}$$

the direction of the ray is same as group velocity. Consider a ray pointing down into the fluid (i.e. one with $k_3 > 0$). Along this ray, $N^2$ decreases and hence $(N^2 - \omega^2)$ decreases; $k_3$decreases. From the equation for group velocity we see that the ray is refracted towards the vertical, as shown in the figure. No wave energy propagates beyond the depth given by N(z)$=\omega$, where both components of group velocity vanish, and where the wave is reflected back up.

# 2 Shallow Water Waves

Shallow-water flow is just one of the many special forms in which hydrodynamics presents itself. Actually, contrary to what the name suggests, the fluid does not have to be water. Certain aspects of flow in the atmosphere are described by the shallow-water equations. The various applications :

- Planetary flows
- Tsunamis
- River flows
- Storm surges
- Tidal flows
- Atmospheric flows : As the density of air is much much less than water, so the effective height of the fluid column is very less.

The basic assumption in shallow-water theory is that any vertical scale H is much smaller than any horizontal scale L. This puts the flows in the class of boundary-layer type flows. How small the H/L ratio should be is not easy to say, but according to some published literature (Le Mehaute, 1976) the upper limit on this ratio is 0.05.

Here I will state the conservation equations in component form to be used in this section.

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + F_x \tag{48}$$

$$\rho\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + F_y \tag{49}$$

$$\rho\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \rho g + F_z \tag{50}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{51}$$

Here we will assume that the fluid is incompressible.

## 2.1 Two-dimensional shallow-water equations

To get the 2-d SWE we have to do a depth integration of the horizontal momentum equations and continuity equation.

Let $a = h + \eta$

$$\int_{z_b}^{a} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + [w]_{z_b}^{a} = 0 \tag{52}$$

$$\frac{\partial}{\partial x}(a\bar{u}) + \frac{\partial}{\partial y}(a\bar{v}) + [w]_{z_b}^{a} - \frac{\partial a}{\partial x}u_s - \frac{\partial a}{\partial y}v_s + \frac{\partial z_b}{\partial x}u_b + \frac{\partial z_b}{\partial y}v_b = 0 \tag{53}$$

The overbar represents a depth average and the surface(s) and bottom(b) correcton terms come in from the interchange of integration and differentiation with the boundary postion depending on x and y. Using the boundary conditions at the top and bottom, the bottom terms cancel and the surface terms yield the rate of change of the surface level.

$$\frac{\partial(\eta)}{\partial t} + \frac{\partial}{\partial x}(a\bar{u}) + \frac{\partial}{\partial y}(a\bar{v}) = 0 \tag{54}$$

Integrating the horizontal momentum equations gives:

$$\frac{\partial}{\partial t}(au) + \frac{\partial}{\partial x}(au^2) + \frac{\partial}{\partial y}(auv) - fav + ga\frac{\partial}{\partial x}a = F_x \tag{55}$$

$$\frac{\partial}{\partial t}(av) + \frac{\partial}{\partial x}(auv) + \frac{\partial}{\partial y}(av^2) - fau + ga\frac{\partial}{\partial y}a = F_y \tag{56}$$

## 2.2 Driving forces

The various type of driving forces are discussed in this section

**1.** Atmospheric pressure gradients

In the driving forces on the right hand side of eqn (55), (56), the atmospheric pressure gradient $\partial p_a/\partial x_i$ may be important for the simulation of storm surges. The value of gradients should be known.

**2.** Wind stress

The wind stress appears as an important driving force in the SWE : the contribution to the right hand side are $(\tau_{sx}, \tau_{sy})/\rho$. The magnitude and direction of the wind stress on the sea surface are determined by the flow in the atmosphere. Usually, the wind speed W is assumed to be known.

**3.** Radiation stress

Wave motions may produce driving forces for the mean flow. There is a net flow in the direction of the wave propagation. If the wave field is known, the radiation stress acts as a driving forces for the mean flow.

**4.** Tidal stress

Oscillating tidal flows can act as driving forces for the equations. They act similar to radiation forcing.

## 2.3 The conserved quantities

The equations (48) - (51) are in the form of local conservation laws, indicating the rate of change of mass or momentum in case of local unbalance of fluxes. Integrating over global domains gives global conservation laws. Here the conserved quantities are just mentioned and for their derivation any standard text on shallow water equations can be referred [15].

**1.** Mass

**2.** Momentum

**3.** Vorticity

$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

**4.** Potential Vorticity

$q = \frac{\zeta + f}{h}$

**5.** Energy

**6.** Potential Enstrophy

The potential enstrophy is defined as $hq^2/2$.

## 2.4 Discussion of the Numerical Scheme

Here I will present the central point of this report. Most of the work was done on developing a model for solution of the SWE using a finite difference scheme.

We have used the following form of the SWE

$$\frac{\partial \bar{V}}{\partial t} + (\frac{f}{\rho} + \beta)\bar{n} \times (p\bar{V}) + \nabla(p + \frac{1}{2}\bar{V}.\bar{V}) = 0 \tag{57}$$

where

$\beta = (\nabla \times \bar{V}).\bar{n}/p$

$p = h + \eta$

h= mean depth

$\eta$= surface elevation

$\bar{n}$ = unit vector normal to the domain

This can be derived by putting in vector identity.

$$v.\nabla v = \frac{1}{2}\nabla(v.v) - v \times (\nabla \times v)$$

### 2.4.1 The finite difference schemes

Sadourny[3] discusses two finite difference schemes : potential enstrophy conservation and energy conservation schemes.

These schemes basically differ in the averaging of the variables over the grids, which gives them these properties. One point to be understood is that the finite difference scheme here are only discretised in space. Once time dimension discretisation is done then the schemes dont conserve any of these quantities. This is discussed in greater detail in [15].

The grid being used is presented in the figure 2

The operators are defined as [3]:

$$\delta_x q(x,y) = \frac{1}{d}[q(x + \frac{d}{2}, y) - q(x - \frac{d}{2}, y)] \tag{58}$$

$$\delta_y q(x,y) = \frac{1}{d}[q(x, y + \frac{d}{2}) - q(x, y - \frac{d}{2})] \tag{59}$$

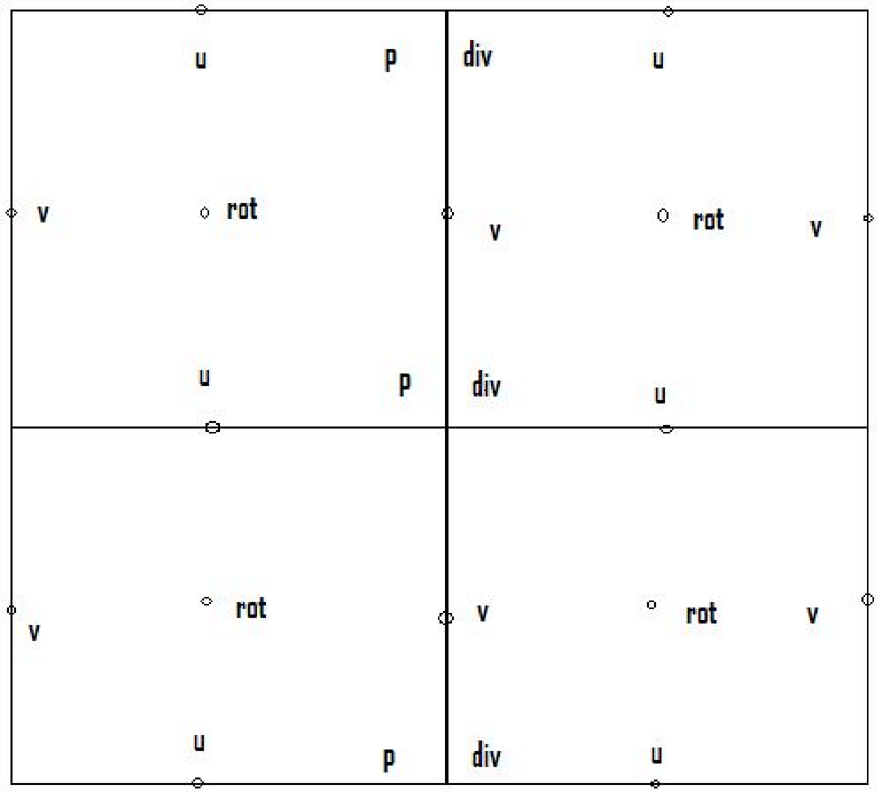


Figure 2: Stagered used for the finite difference scheme

$$\bar{q}^x(x,y) = \frac{1}{2}[q(x+\frac{d}{2},y) + q(x-\frac{d}{2},y)] \tag{60}$$

$$\bar{q}^y(x,y) = \frac{1}{2}[q(x,y+\frac{d}{2}) + q(x,y-\frac{d}{2})] \tag{61}$$

The mass fluxes U and V are defined at the local points where the velocity components u and v are located.

$$U = \bar{P}^x u \tag{62}$$

$$V = \bar{P}^y v \tag{63}$$

Defininining another quantity H, this quantity is located where the pressure is defined.

$$H = P + \frac{1}{2}(\overline{u^2}^x + \overline{v^2}^y) \tag{64}$$

Potential vorticity is located at the mesh centers and defined as

$$\eta = \frac{(\delta_x v - \delta_y u) + f}{\bar{\bar{P}^x}^y} \tag{65}$$

Expression for total mass, energy and absolute potential enstrophy :

$$M = \sum P \tag{66}$$

$$E = \frac{1}{2}\sum(P^2 + P\bar{u^2}^x + P\bar{v^2}^y) \tag{67}$$

$$Z = \frac{1}{2}\sum \eta^2 \bar{\bar{P}^x}^y \tag{68}$$

Note that the derivation operators $\delta_x$, $\delta_y$ are skew symmetric linear operators, and further that the averaging operators $(\bar{\ }^x, \bar{\ }^y)$ are symmetric linear operators, i.e.,

$$\sum a\bar{b}^x = \sum b\bar{a}^x \tag{69}$$

$$\sum a\delta_x b = -\sum b\delta_x a \tag{70}$$

The time derivative of total energy reads

$$\frac{dE}{dt} = \sum (U\frac{\partial u}{\partial t} + V\frac{\partial v}{\partial t} + H\frac{\partial P}{\partial t}) \tag{71}$$

A simple energy conserving model is defined as[3]

$$\frac{\partial u}{\partial t} - \overline{\eta\overline{V}^{x}}^{y} + \delta_x H = 0 \tag{72}$$

$$\frac{\partial v}{\partial t} - \overline{\eta\overline{U}^{y}}^{x} + \delta_y H = 0 \tag{73}$$

$$\frac{\partial P}{\partial t} + \delta_x U + \delta_y V = 0 \tag{74}$$

So from the total energy derivative (71) we have

$$\begin{aligned} &\tfrac{dE}{dt} + \sum (V\overline{\eta\overline{U}^{y}}^{x} - U\overline{\eta\overline{V}^{x}}^{y} \\ &+ \sum (U\delta_x H + H\delta_x U) \\ &+ \sum (U\delta_y H + H\delta_y V) = 0 \end{aligned} \tag{75}$$

where each of the summations cancel due to the symmetry or skew symmetry properties of the operators.

The potential enstrophy conservation scheme:

$$\frac{\partial u}{\partial t} - \overline{\eta}^{y}\overline{\overline{V}^{x}}^{y} + \delta_x H = 0 \tag{76}$$

$$\frac{\partial v}{\partial t} + \overline{\eta}^{x}\overline{\overline{U}^{y}}^{x} + \delta_y H = 0 \tag{77}$$

$$\frac{\partial P}{\partial t} + \delta_x U + \delta_y V = 0 \tag{78}$$

The corresponding vorticity equation is:

$$\frac{\partial}{\partial t}(\eta\overline{\overline{P}^{x}}^{y}) + \delta_x(\overline{\eta}^{x}\overline{\overline{U}^{y}}^{x}) + \delta_y(\overline{\eta}^{y}\overline{\overline{V}^{x}}^{y}) = 0 \tag{79}$$

which, when combined with the averaged continuity equation

$$\frac{\partial}{\partial t}(\overline{\overline{P}^{x}}^{y}) + \delta_x(\overline{\overline{U}^{y}}^{x}) + \delta_y(\overline{\overline{V}^{x}}^{y}) = 0 \tag{80}$$

yields the conservative potential enstrophy equations

$$\frac{\partial}{\partial t}(\eta^2\overline{\overline{P}^{x}}^{y}) + \delta_x(\tilde{\eta^2}^{x}\overline{\overline{U}^{y}}^{x}) + \delta_y(\tilde{\eta^2}^{y}\overline{\overline{U}^{x}}^{y}) = 0 \tag{81}$$

This scheme does not conserve energy except in the case of pure rotational motion.

## 2.5 Notes on Code

For a detailed discussion of the code, the code documentation should be referred. Here only a summary is given.

The model was not developed from scratch. An existing code called the flexible modeling system-shallow water model was used. This code was written by Dr. Ronald Pacanowski and Dr. Zhi Liang at Geophysical Fluid Dynamics Laboratory, Princeton. The original code was written for the potential enstrophy conservation scheme using the filtered leap frog scheme.

The code uses flexible modeling system, which is a set of APIs written in FORTRAN to work over the MPI libraries. The FMS takes care of the parallel processing, splitting into domain and managing the computational and data domains, and the interactions between the different processors.

Taking the original code as an example, I wrote the energy conservation scheme into the code. I also changed the time stepping scheme to Adams-Bashforth scheme [15].

The code is written in FORTRAN 90. The output is written in Netcdf files, which can be plotted using Ferret or MATLAB.

The code and the documentation can be downloaded from : "http://dhruv.lifeasiknow-it.com/codes.html"

## 2.6 Discussion of results

### 2.6.1 Sumatra 2004 tsunami

The 2004 Indian Ocean earthquake was an undersea megathrust earthquake(9.3 that occurred at 00:58:53 UTC on December 26, 2004, with an epicentre off the west coast of Sumatra, Indonesia. The quake itself is known by the scientific community as the Sumatra-Andaman earthquake. The resulting tsunami itself is given various names, including the 2004 Indian Ocean tsunami, Asian Tsunami, Indonesian Tsunami, and Boxing Day Tsunami.

I used the FMS-SWM, with the enstrophy conservation scheme to simulate the tsunami. The code was run with the following parameters:

time step = 1s

gird size = 440 X 240 (30E to 140E, 30S to 30N)

The initial conditions for the displacement of the ocean surface were hand written into a netcdf file; they are not exact. It was set that there was positive lift of 4m and negative bump of -2 m formed at

| Station | Coordinates | Time(obs.) | Time(mod.) | Height(obs.) | Height(mod.) |
|---|---|---|---|---|---|
| Cocos Islands | 12.13°S; 96.88°E | 03:17 | 3:18 | 59cm | 70cm |
| Colombo | 06.93°N; 79.83°E | 03:49 | 3:43 | 300cm | 45-50cm |
| Male | 04.18°N; 73.52°E | 04:14 | 4:28 | 215cm | 60cm |
| La Reunion | 20.92°S; 55.30°E | 07:55 | 7:58 | 70cm | 9cm |

Table 1: Comparison of tide-gauge readings and model outputs

the time of earthquake.

The simulation was run for 14 hours and the data was written to a Netcdf file every 5 mins. The results are presented in the the figure 3.

The numerically simulations were compared with the tide guage readings given in [14] .

The tide guage readings are presented here for the stations which were compared.

*All these stations first saw the crests.

The model out puts have been presented in the table 1.

- Arrival Times

Looking at the results we can conclude that the time of arrival obtained from the model output are very well in correspondence with the observed tide gauge readings. The arrival time is correct as the wave propagation speed, $c = \sqrt{g(h+\eta)}$, is function of ocean depth. The effect of the wave height is negligble compared to the depth.

- Arrival Heights

The arrival heights from the numerical model are nowhere in agreement with the observed readings. The main reason for this is that the initial condition that was specified are not real. It is assumed that the water displacement by the displacement of the bottom is the same as the earth that was shifted. Wheras this is not true as the water is a continuous medium and doesnot displace the same way as solid earth. The arrival height will be correctly predicted, if the initial condition is put in properly.

### 2.6.2 Continental Slope

A case was tested in which the effect of the change in depth at the continental slope was tested on

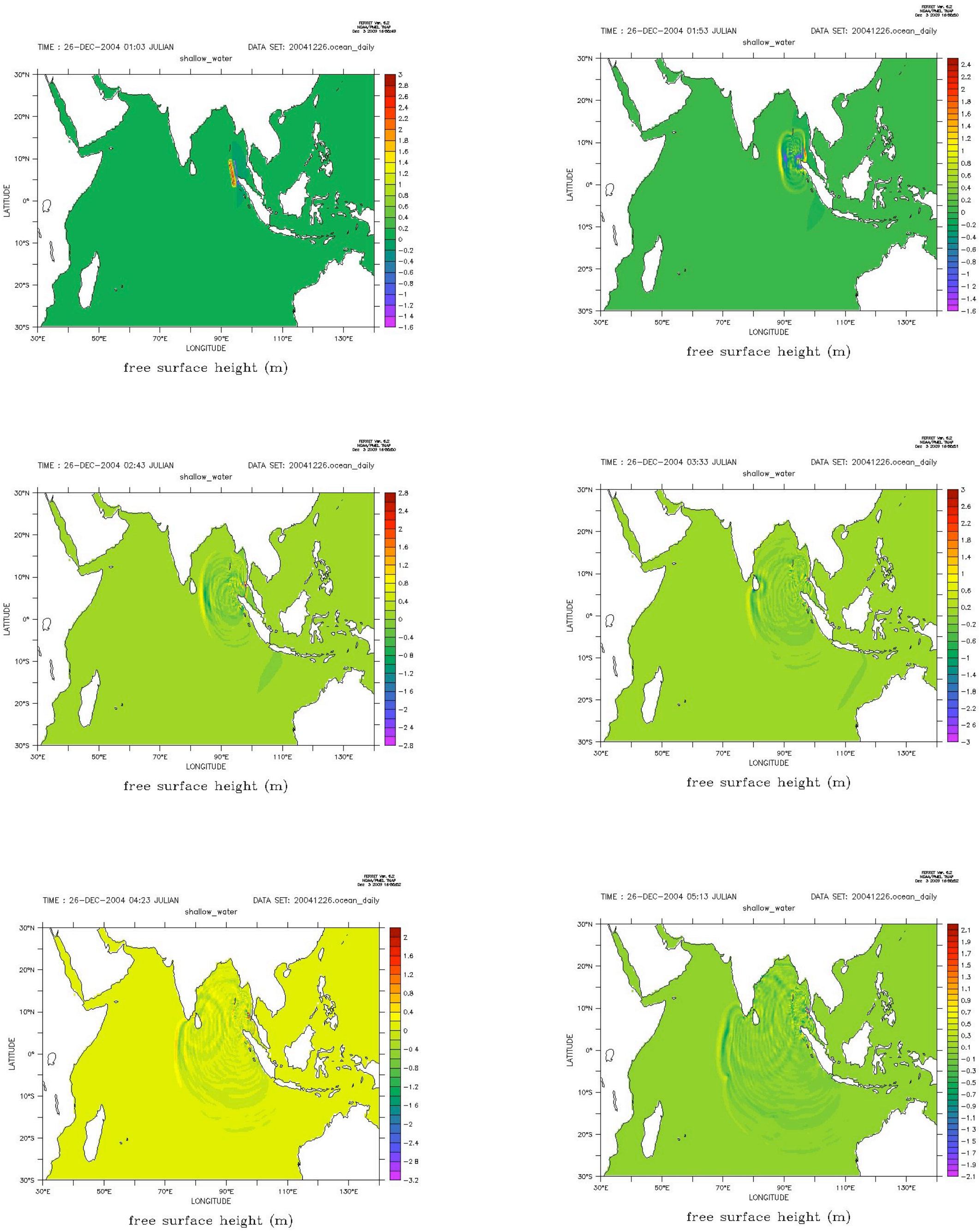


Figure 3: Sumatra Tsunami 2004 simulations

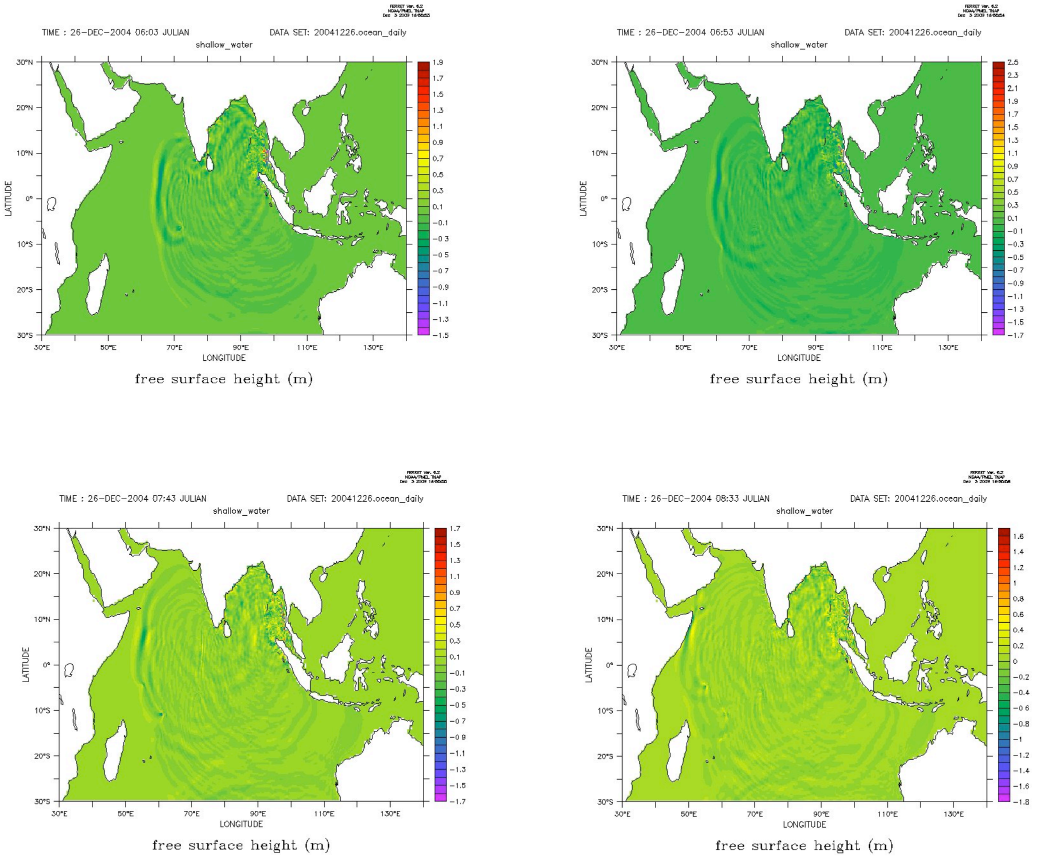


Figure 4: Sumatra Tsunami 2004 Simulation

the wave amplitudes that passed the slope. Two basic experiments were done to see the difference in amplifications; in one the wavelength of the incoming wave was varried keeping the slope constant and in the other the slope was varried keeping the wavelength constant.

The depth of the ocean changed from 2000m at the start of the slope to 200m at the end. A linear profile was assumed for the slope.

Also all the simulations were done with the coriolis force set to zero.

- Wavelength variation

Te wavelength was varried from 720km to 14200km. The 720km is approximately the wavelength of a tsunami and 14200km represents Rossby waves. The Wavelengths in between can very well represent tides. There is not any clear distinction on the wavelengths of these processes, but the order of magnitude is approximately in this range. The depth decresed from 2000m to 200m over length of 1100km. The angle of the slope being 0.1 degree. As the model runs on the speherical coordinate system, the ideal case was defined in terms of latitude and longitude. The slope started at 170°E and ended at 180°E. The forcing was done at 100°E.

| Wavelength | Frequency | Amplification |
|---|---|---|
| 720km | 2e-4 | 1.825 |
| 1410km | 1e-4 | 1.775 |
| 7200km | 2e-5 | 1.64 |
| 14200km | 1e-5 | 1.55 |

Table 2: Wavelength vs. amplification

The amplification is defined as the ratio of wave height after the slope is crossed to the waveheight of the wave reaching the slope.

We clearly see an incerease in amplification as the wave length decreases and the frequency increases. This observation is something that is expected. The rate at which the wave energy comes and starts to move up the slope, will depend on both the wave celerity and the wave speed. The wave speed is a function of ocean depth($c = \sqrt{g(h+\eta)}$). As the depth decreases the wave speed decreases. The wave energy flowing in from the deeper ocean is constant and faster, but the wave energy flowing out is much slower. This unbalance in the inflow and outflow of energy leads to amplification of wave height(potential energy increases to conserve total energy). When the frequency is higher the rate of energy inflow is also faster, hence the amplification is more for greater frequencies.

- Slope Angle Variation

The incoming wave had frequency of 1e-4 and wavelegth of 1410km. The incoming wave had an amplitude or 0.2m.

| Angle | Propagated Wave height | Reflected Wave height |
|---|---|---|
| $1^{\circ}$ | 0.325m | 0.09m |
| $0.1^{\circ}$ | 0.350m | 0.01m(+), 0.04m(-) |

Table 3: Angle Variation

The amplification of the propagated wave increases as the angle decreases. Also the reflected wave has maximum amplitude for the case of greatest slope. The total energy needs to be conserved. Thus when the reflected wave has a greater amplitude, hence greater energy the propagated wave has lesser amplification.

The reflected wave showed an interesting characterstic. The reflected wave was reflected in parts, the crest was reflected when then the wave reached the start of the slope. Wheras the trough of the reflected wave was reflected when the propeagating wave reached the end of the slope. These reflections maybe be due to numerical problems created as the slope is considered to be piecewise topography and the corners at the start and end of the slope might be responsible for false reflections.

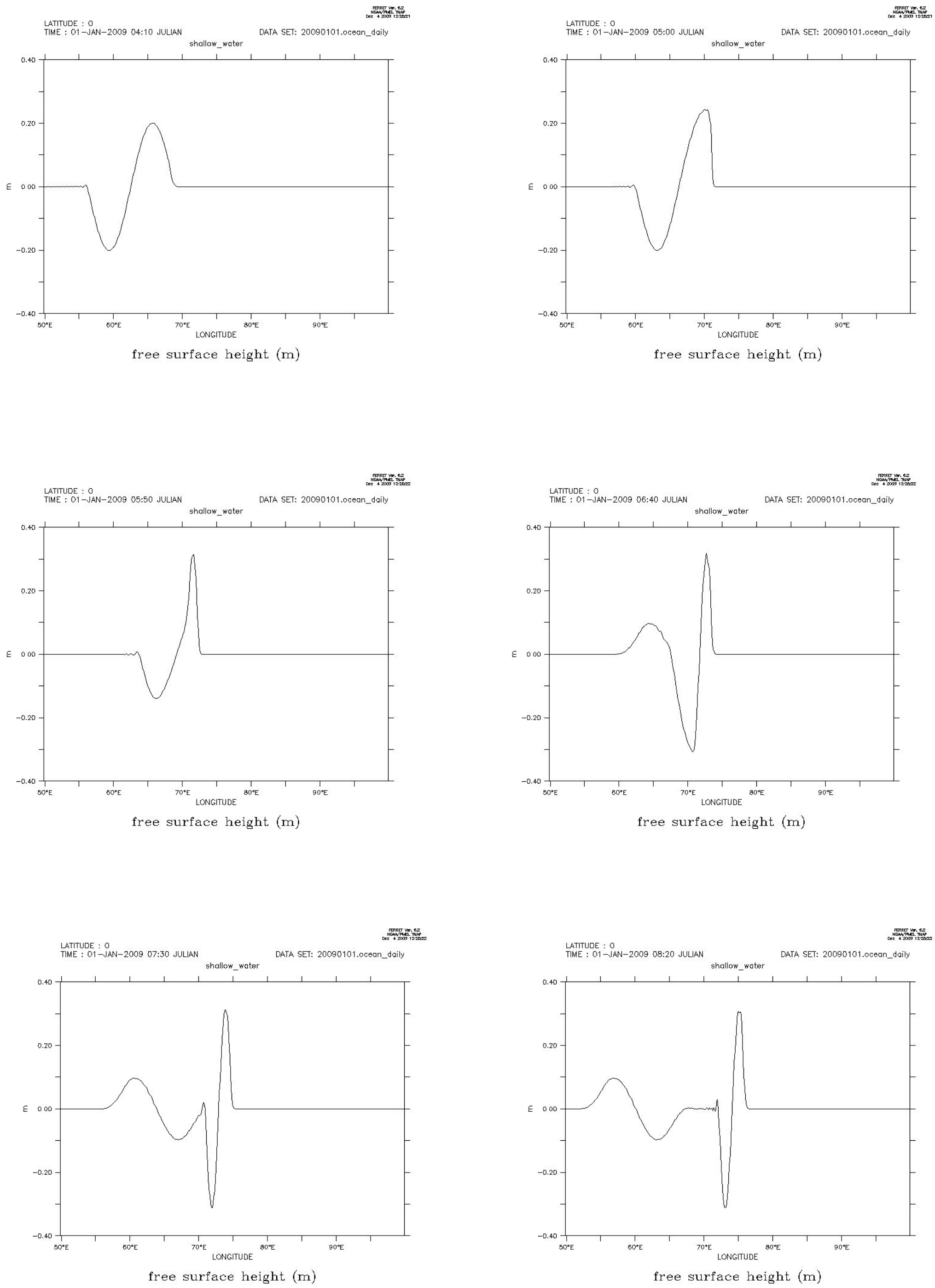


Figure 5: Continental Slope Angle 1.0°

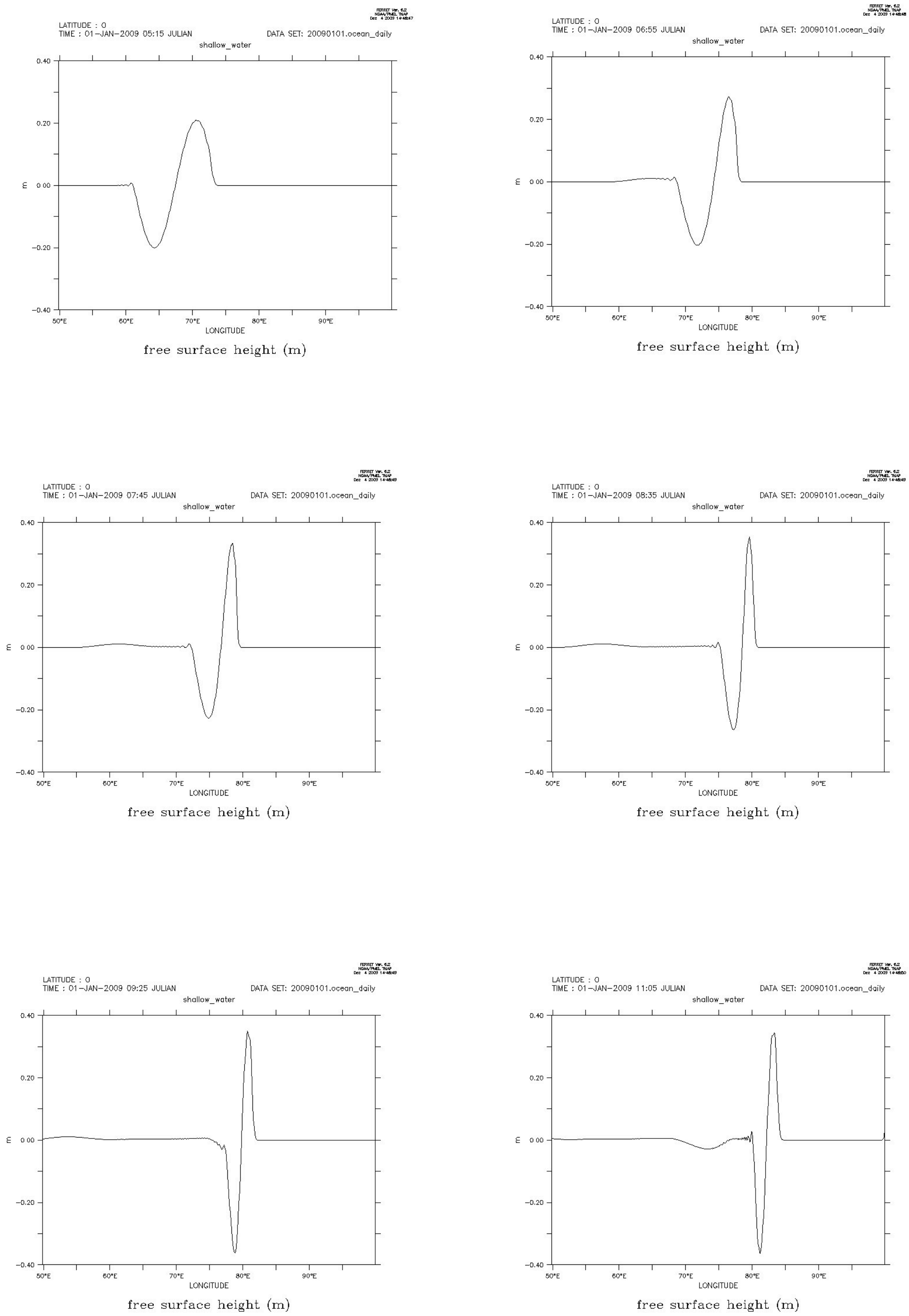


Figure 6: Continental Slope Angle 0.1°

## 3 Conclusions

A model to solve the shallow water equations was developed. This model was used to solve initial value standard test cases. The model was validated using the Sumatra tsunami case and it was found that the model gave the correct arrival times. There was a problem in the arrival heights but that is probably a problem with the initial condition rather than the model numerics. The numerical results for the continental slope were logically correct. The linearised equations can be solved and compared with the numerical results.

The dynamics of the internal waves was studied, and their reflection from slopes in the coastal bathymetry and refraction due to variation in the Brunt Vaisala frequency in the ocean was studied. This has given me insight into this phenomena and made me ready to explore them in greater depth using numerical models or observational data.

## 4 Future Scope

For the case of internal waves a non-hydrostatic numerical model must be used to simulate the reflection and refraction processes. A few suggested models would be: MIT General Circulation Model or The University of Waterloo-Internal Gravity Wave Model. Numerical simulations of the internal waves will provide insight into the mixing and energy transfer that takes place as an effect of their interaction with slopes.

For the case of the FMS-Shallow Water model that has been developed there are still a lot of numerical imporovements that can be done. The land boundaries need to be handled in a better way, and the work done in this direction by Ketafian[15] can be incorportated into the model. The discretisation scheme can be changed to a scheme that conserves both energy and potential enstophy, such as the one presented by Arakawa and Lamb. For the individual cases that were run the initial conditions can be improved. For the continental slope case run using the model, an analytical analysis needs to be done to check the validity of the results.

## List of Figures

# 5 Appendix

## 5.1 Ray Theory

A propagating wave is a travelling disturbance about an equilibrium state; it may be imagined as a moving surface or front, on which some physical quantity retains a coherent contrast with respect to its surroundings. The general wave forms is given by

$$\phi(\bar{x}, t) = A(\bar{x}, t)exp[iS(\bar{x}, t)] \tag{82}$$

The variation of $A(\bar{x}, t)$ is assumed to be slowly varying compared to the phase function $S(\bar{x}, t)$. Such slowly modulating waves are called nearly plane waves, and this is called the WKB approximation.

The wavenumber and the frequency are given by

$\bar{k} = \nabla S$, $\omega = -\frac{\partial S}{\partial t}$

This lead to the conservation of crests equation

$$\frac{\partial \bar{k}}{\partial t} + \nabla \omega = 0 \tag{83}$$

The rate of progression of a surface of constant phase $S(\bar{x}, t)$=constant is found by letting $dS = 0$, which gives

$$\frac{\partial S}{\partial t} dt + \nabla S.d\bar{x} = 0 \tag{84}$$

This gives us $\omega = \bar{k}.\bar{c}$

In a general case we get a dispersion relation like

$$\omega(\bar{x}, t) = \sigma[\bar{k}(\bar{x}, t); \lambda(\bar{x}, t)] \tag{85}$$

where $\lambda$ accounts for the variations in the medium.

$$\frac{\partial k_i}{\partial t} + \frac{\partial \sigma}{\partial k_j}\frac{\partial k_j}{\partial x_i} + \frac{\partial \sigma}{\partial \lambda}\frac{\partial \lambda}{\partial x_i} = 0 \tag{86}$$

As K is the gradient of a scalar, it implies that $\nabla \times \bar{k} = 0$, we have $\partial k_j/\partial x_i = \partial k_i/\partial x_j$. If we then describe the group velocity by the relation

$$\bar{c}_{gi} = \partial \sigma/\partial k_i \tag{87}$$

Eqn (86) may be rewritten as

$$\frac{\partial \bar{k}}{\partial t} + \bar{c}_g.\nabla \bar{k} = -\frac{\partial \sigma}{\partial \lambda}\nabla \lambda \tag{88}$$

Similarily, differentiating eqn (85) with respect to t and using eqn (87), we obtain

$$\frac{\partial \omega}{\partial t} + \bar{c}_g.\nabla \omega = \frac{\partial \sigma}{\partial \lambda}\frac{\partial \lambda}{\partial t} \tag{89}$$

Eqns (88), (89) have the same characterstic curves, which are obtained by integrating the relationships

$$\frac{d\bar{x}}{dt} = \bar{c}_g = \frac{\partial \sigma}{\partial \bar{k}} \tag{90}$$

These characterstics are called rays. Integration of eqn (90) gives

$$\bar{x} - \int \bar{c}_g dt = \alpha \tag{91}$$

where $\alpha$ is a constant vector which varies from one ray to the next. The time derivative along a ray, i.e. keeping $\alpha$ fixed, is then

$$\left(\frac{d}{dt}\right)_{\alpha} = \frac{\partial}{\partial t} + \bar{c}_g.\nabla \tag{92}$$

Given some initial ad boundary conditions and an additional statement on variation of wave amplitude along a ray, one can construct the wave solution over the rest of the space-time domain in which the ray theory is applicable.

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